METALOGIC is the study of formal systems (e.g., our propositional system) and the attempt to prove things about these systems.

1. There is no longest propositional wff.
2. We can define "$\vee$," "$\supset$," and "$\equiv$" using "$\bullet$" and "$\sim$."
3. Our propositional system is *sound*: every provable argument is valid (on the truth-table test).
4. Our propositional system is *complete*: every valid argument is provable.

We can define "∨," "⊃," and "≡" using "•" and "~."

$(P \vee Q)$	=	$\sim(\sim P \cdot \sim Q)$
At least one is true	=	Not both are false.
$(P \supset Q)$	=	$\sim(P \cdot \sim Q)$
If P then Q	=	We don't have P true and Q false.
$(P \equiv Q)$	=	$(\sim(P \cdot \sim Q) \cdot \sim(Q \cdot \sim P))$
P if and only if Q	=	We don't have P true and Q false, and we don't have Q true and P false.

Similarly, all the connectives can be defined using "~" and "∨," "~" and "⊃," or NAND.

NAND: "(P | Q)" = "not both A and B"

Some alternative notations:

ENGLISH	SYMBOLS
Not-A	$\sim$A, -A, $\neg$A, Na
Both A and B	(A $\cdot$ B), (A & B), (A $\wedge$ B), Kab
Either A or B	(A $\vee$ B), Aab
If A then B	(A $\supset$ B), (A $\rightarrow$ B), Cab
A if and only if B	(A $\equiv$ B), (A $\leftrightarrow$ B), Eab

Page 336–338

Our propositional system is sound: every provable argument is valid (on the truth-table test).

To show this, we basically need to show that our S- and I-rules are truth-preserving and that RAA is truth-preserving.

1 … = 1 ⇦
2 … = 1 ⇦ Suppose that all these are true.

asm: ~A
…
∴ B
∴ ~B We derive a contradiction.

∴ A 🡐 Does A then have to be true?

A THEOREM is a wff that is provable from zero premises; "$(P \vee \sim P)$" is an example of a theorem.

$$
\begin{array}{lll}
 & [\ \therefore\ (P \vee \sim P) & \text{Valid} \\
* & 1 \quad \text{asm: } \sim(P \vee \sim P) & \\
 & 2 \quad \therefore\ \sim P \ \ \{\text{from } 1\} & \\
 & 3 \quad \therefore\ P \ \ \{\text{from } 1\} & \\
 & 4 \quad \therefore\ (P \vee \sim P) \ \ \{\text{from } 1;\ 2 \text{ contra } 3\} &
\end{array}
$$

Our propositional system is *consistent*:
no formula and its denial are both theorems.

From the soundness proof, it follows that all theorems have all-1 truth tables. But no formula and its denial both have all-1 truth tables. Therefore, no formula and its denial are both theorems.

Our system is *complete*: any valid propositional argument is provable.

Start	S&I	RAA	Assume	Refute

Assume that we correctly apply our PROOF STRATEGY; then:

We'll end in the RAA step with all assumptions blocked off, end in the REFUTE step, or keep going endlessly.
If we end in the RAA step with all assumptions blocked off, then we'll get a proof.
If we end in the REFUTE step, then the argument is invalid.
We won't keep going endlessly.
∴ If the argument is valid, then we'll get a proof.

Premise 3 is true because wffs of the 9 complex forms will dissolve into smaller parts and eventually into simple wffs, the larger forms are true if the smaller parts are true, and the simple wffs we end up with are consistent and thus give truth conditions making all the previous wffs true – thus making the premises of the original argument true while its conclusion is false.

Pages 340–343

The first propositional proof systems were *axiomatic* (as opposed to *inferential*). Here a PROOF is a vertical sequence of zero or more premises followed by one or more derived steps, where each derived step is an axiom or follows from earlier lines by the inference rule or a substitution of definitional equivalents. In Bertrand Russell's system, these hold for any wffs "A," "B," and "C":

Axioms	1. $((A \vee A) \supset A)$ 2. $(A \supset (A \vee B))$ 3. $((A \vee B) \supset (B \vee A))$ 4. $((A \supset B) \supset ((C \vee A) \supset (C \vee B)))$
Inference Rule	$(A \supset B),\ A \ \rightarrow\ B$
Definitions ("$\vee$" and "$\sim$" are basic)	1. $(A \supset B) \ = \ (\sim A \vee B)$ 2. $(A \cdot B) \ = \ \sim(\sim A \vee \sim B)$ 3. $(A \equiv B) \ = \ ((A \supset B) \cdot (B \supset A))$

Axiomatic proofs are difficult; here's a proof of "(P ∨ ~P)":

1 ∴ (((P ∨ P) ⊃ P) ⊃ ((~P ∨ (P ∨ P)) ⊃ (~P ∨ P))) {from axiom 4, substituting "(P ∨ P)" for "A," "P" for "B," and "~P" for "C"}
2 ∴ ((P ∨ P) ⊃ P) {from axiom 1, substituting "P" for "A"}
3 ∴ ((~P ∨ (P ∨ P)) ⊃ (~P ∨ P)) {from 1 and 2}
4 ∴ (P ⊃ (P ∨ P)) {from axiom 2, substituting "P" for "A" and "P" for "B"}
5 ∴ (~P ∨ (P ∨ P)) {from 4, substituting things equivalent by definition 1}
6 ∴ (~P ∨ P) {from 3 and 5}
7 ∴ ((~P ∨ P) ⊃ (P ∨ ~P)) {from axiom 3, substituting "~P" for "A" and "P" for "B"}
8 ∴ (P ∨ ~P) {from 6 and 7}