$(\sim B \supset (A \cdot C))$
$(A \supset \sim C)$
$\therefore B$

This is our sample argument.

Formal Proofs

From now on, formal proofs will be our main way to test arguments. We’ll begin with easier proofs. Our initial strategy for constructing proofs has three steps.

1	$(\sim B \supset (A \cdot C))$		
2	$(A \supset \sim C)$		
[∴ B		[	Block off conclusion
3	asm: $\sim B$	←	Assume the opposite

Step 1: START

Block off the conclusion and add “asm:” followed by the conclusion’s simpler contradictory.

1 **(~B ⊃ (A · C))**
2 **(A ⊃ ~C)**
[∴ B
3 asm: ~B

Here the complex wffs are 1 and 2, both **IF-THENs**. You can infer from these if you have the first part true or the second false.

Step 2: S&I

Begin the S&I step by glancing at the complex wffs and noticing their forms. You can simplify **AND**, **NOR**, and **NIF** – and you can infer with **NOT-BOTH**, **OR**, and **IF-THEN** if certain other wffs are available.

*	1	**(~B ⊃ (A · C))**		
	2	(A ⊃ ~C)		
	[∴ B			
	3	asm: **~B**		
	4	∴ **(A · C)**	{from 1 and 3}	←

IF-THEN rule:

(~B ⊃ (A · C))
~B
———————
(A · C)

Step 2: S&I

Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using S- and I-rules. Star any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

*	1	(~B ⊃ (A • C))	
	2	(A ⊃ ~C)	
	[∴ B		
	3	asm: ~B	
*	4	∴ **(A • C)** {from 1 and 3}	
	5	∴ **A** {from 4}	🡐
	6	∴ **C** {from 4}	🡐

AND rule:

$$\frac{(A \cdot C)}{A, C}$$

Step 2: S&I

Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using S- and I-rules. Star any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

*	1	(~B ⊃ (A • C))	
*	2	**(A ⊃ ~C)**	
	[∴ B		
	3	asm: ~B	
*	4	∴ (A • C)	{from 1 and 3}
	5	∴ **A**	{from 4}
	6	∴ C	{from 4}
	7	∴ **~C**	{from 2 and 5} ←

IF-THEN rule:

(A ⊃ ~C)
A
———
~C

Step 2: S&I

Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using S- and I-rules. Star any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

*	1	(~B ⊃ (A · C))	
*	2	(A ⊃ ~C)	
		[∴ B	
	3	┌ asm: ~B	
*	4	│ ∴ (A · C) {from 1 and 3}	
	5	│ ∴ A {from 4}	
	6	│ ∴ **C** **{from 4}**	
	7	└ ∴ **~C** **{from 2 and 5}**	
	8	∴ B {from 3; 6 contradicts 7}	

Valid

block off from the assumption on down

← derive conclusion

Step 3: RAA

> When some pair of not-blocked-off lines contradicts, apply RAA and derive the original conclusion. Your proof is done!

S- and I-Rules

AND

$$\frac{(P \cdot Q)}{P,\ Q}$$

NOR

$$\frac{\sim(P \vee Q)}{\sim P,\ \sim Q}$$

NIF

$$\frac{\sim(P \supset Q)}{P,\ \sim Q}$$

NOT-BOTH

$$\frac{\sim(P \cdot Q) \quad P}{\sim Q} \qquad \frac{\sim(P \cdot Q) \quad Q}{\sim P}$$

affirm one part

OR

$$\frac{(P \vee Q) \quad \sim P}{Q} \qquad \frac{(P \vee Q) \quad \sim Q}{P}$$

deny one part

IF-THEN

$$\frac{(P \supset Q) \quad P}{Q} \qquad \frac{(P \supset Q) \quad \sim Q}{\sim P}$$

affirm 1st or deny 2nd

S- and I-Rules

AND: $\dfrac{(P \cdot Q)}{P,\ Q}$	NOR: $\dfrac{\sim(P \vee Q)}{\sim P,\ \sim Q}$	NIF: $\dfrac{\sim(P \supset Q)}{P,\ \sim Q}$
NN: $\dfrac{\sim\sim P}{P}$	IFF: $\dfrac{(P \equiv Q)}{(P \supset Q),\ (Q \supset P)}$	NIFF: $\dfrac{\sim(P \equiv Q)}{(P \vee Q),\ \sim(P \cdot Q)}$
NOT-BOTH: $\dfrac{\sim(P \cdot Q),\ P}{\sim Q}$ $\dfrac{\sim(P \cdot Q),\ Q}{\sim P}$	OR: $\dfrac{(P \vee Q),\ \sim P}{Q}$ $\dfrac{(P \vee Q),\ \sim Q}{P}$	IF-THEN: $\dfrac{(P \supset Q),\ P}{Q}$ $\dfrac{(P \supset Q),\ \sim Q}{\sim P}$

RAA: Suppose that some pair of not-blocked-off lines has contradictory wffs. Then block off all the lines from the last not-blocked-off assumption on down and infer a line consisting in "∴" followed by a contradictory of that assumption.

*	1	(~B ⊃ (A · C))	←	premises (no "asm" or "∴")
*	2	(A ⊃ ~C)	←	
		[∴ B		
	3	┌ asm: ~B	←	assumption ("asm")
*	4	│ ∴ (A · C) {from 1 and 3}	←	derived lines ("∴")
	5	│ ∴ A {from 4}	←	
	6	│ ∴ C {from 4}	←	
	7	└ ∴ ~C {from 2 and 5}	←	
	8	∴ B {from 3; 6 contradicts 7}	←	

A *formal proof* is a vertical sequence of zero or more premises followed by one or more assumptions or derived lines, where each derived line follows from previously not-blocked-off lines by one of the S- and I-rules listed above or by RAA, and each assumption is blocked off using RAA.

Two wffs are *contradictories* if they are exactly alike except that one starts with an additional "~."

A *simple wff* is a letter or its negation; any other wff is *complex*.

Valid

```
* 1       (~B ⊃ (A • C))
* 2       (A ⊃ ~C)
       [ ∴ B
  3    ┌ asm: ~B
* 4    │ ∴ (A • C)   {from 1 and 3}
  5    │ ∴ A   {from 4}
  6    │ ∴ C   {from 4}
  7    └ ∴ ~C   {from 2 and 5}
  8  ∴ B   {from 3; 6 contradicts 7}
```

Proof Strategy

1 START: Assume the opposite of the conclusion.

2 S&I: Derive whatever you can using S- and I-rules, until you get a contradiction.

3 RAA: Apply RAA and derive the original conclusion.

	1	$(A \supset B)$	
		$[\therefore (B \supset A)$	
*	2	asm: $\sim(B \supset A)$	
	3	$\therefore B$ {from 2}	We can derive
	4	$\therefore \sim A$ {from 2}	nothing further.

Proof strategy to include invalid arguments:

1 START: Assume the opposite of the conclusion.

2 S&I: Derive whatever you can using S- and I-rules.

3 RAA: If you get a contradiction, apply RAA and derive the original conclusion.

4 REFUTE: If you don't get a contradiction, construct a refutation box.

1	$(A^0 \supset B^1) = 1$	
	$[\therefore (B^1 \supset A^0) = 0$	
* 2	asm: $\sim(B \supset A)$	
3	$\therefore B$	{from 2}
4	$\therefore \sim A$	{from 2}

Invalid

B, ~A

Step 4 – REFUTE: If you can't get a contradiction, then:

- draw a box containing any simple wffs (letters or their negation) that aren't blocked off;
- in the original argument, mark each letter "1" or "0" or "?" depending on whether you have the letter or its negation or neither in the box;
- if these truth conditions make the premises all true and conclusion false, then this shows the argument to be invalid.

*	1	(~B ⊃ (A • C))	Valid
*	2	(A ⊃ ~C)	
		[∴ B	
	3	┌ asm: ~B	
*	4	│ ∴ (A • C) {from 1 and 3}	
	5	│ ∴ A {from 4}	
	6	│ ∴ C {from 4}	
	7	└ ∴ ~C {from 2 and 5}	
	8	∴ B {from 3; 6 contradicts 7}	

	1	$(A^0 \supset B^1) = 1$	Invalid
		[∴ $(B^1 \supset A^0) = 0$	B, ~A
*	2	asm: ~(B ⊃ A)	
	3	∴ B {from 2}	
	4	∴ ~A {from 2}	

1. START: Assume the opposite of the conclusion.
2. S&I: Derive whatever you can using S- and I-rules.
3. RAA: If you get a contradiction, apply RAA and derive the original conclusion.
4. REFUTE: If you don't get a contradiction, construct a refutation box.

1 (B ∨ A)
2 (B ⊃ A)
[∴ ~(A ⊃ ~A)
3 asm: (A ⊃ ~A) ← We're stuck!

Here we get stuck using our old strategy –
so we need to make another assumption.

1 START: Assume the opposite of the conclusion.
2 S&I: Derive whatever you can using S- and I-rules.
3 RAA: If you get a contradiction, apply RAA and derive the original conclusion.
4 REFUTE: If you don't get a contradiction, construct a refutation box.

1 (B ∨ A)
2 (B ⊃ A)
[∴ ~(A ⊃ ~A)
3 asm: (A ⊃ ~A) ← We're stuck!

We're stuck when:

We can't apply S- or I-rules further.

And we can't prove the argument VALID (since we have no contradiction) or INVALID (since we don't have enough simple wffs for a refutation).

1 (B ∨ A)
2 (B ⊃ A)
[∴ ~(A ⊃ ~A)
3 asm: (A ⊃ ~A)
4 asm: B {break up 1} ← When you're stuck, try to make another assumption.

ASSUME: Look for a complex wff that isn't starred or blocked off or broken. This wff will have one of these forms:

NOT-BOTH	~(A • B)
OR	(A ∨ B)
IF-THEN	(A ⊃ B)

Assume one side or its negation – and then return to step 2 (S&I).

1 (B ∨ A)

** 2 (B ⊃ A)

[∴ ~(A ⊃ ~A)

** 3 asm: (A ⊃ ~A)

4 asm: B {break up 1}

5 ∴ A {from 2 and 4} ← Contradiction!

6 ∴ ~A {from 3 and 5} ←

S&I: Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using S- and I-rules. Star (*with one star for each live assumption*) any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

1 (B ∨ A)

2 (B ⊃ A)

[∴ ~(A ⊃ ~A)

3 asm: (A ⊃ ~A)

4 ┌ asm: B {break up 1}

5 │ **∴ A {from 2 and 4}**

6 └ **∴ ~A {from 3 and 5}**

7 ∴ ~B {from 4; 5 contradicts 6} ← Apply RAA.

RAA: If you have a contradiction, apply RAA on the last live assumption. If all assumptions are now blocked off, you've proved the argument valid. *Otherwise, erase star strings having more stars than the number of live assumptions* – and then return to step 2 (S&I).

* 1 (B ∨ A)

2 (B ⊃ A)

[∴ ~(A ⊃ ~A)

* 3 ┌ asm: (A ⊃ ~A)

4 │ ┌ asm: B {break up 1}

5 │ │ ∴ A {from 2 and 4}

6 │ └ ∴ ~A {from 3 and 5}

7 │ ∴ ~B {from 4; 5 contradicts 6}

8 │ **∴ A {from 1 and 7}** ←

9 └ **∴ ~A {from 3 and 8}** ←

10 ∴ ~(A ⊃ ~A) {from 3; 8 contradicts 9} ←

Valid

We use "~B" to get a contradiction & finish the proof.

*	1	(B ∨ A)	Valid
	2	(B ⊃ A)	
		[∴ ~(A ⊃ ~A)	
*	3	┌ asm: (A ⊃ ~A)	
	4	│ ┌ asm: B {break up 1}	
	5	│ │ ∴ A {from 2 and 4}	
	6	│ └ ∴ ~A {from 3 and 5}	
	7	│ ∴ ~B {from 4; 5 contradicts 6}	
	8	│ ∴ A {from 1 and 7}	
	9	└ ∴ ~A {from 3 and 8}	
	10	∴ ~(A ⊃ ~A) {from 3; 8 contradicts 9}	

Strategy:

Start
S&I
RAA
Assume
Refute

1 ~(A · B)

[∴ (~A · ~B)

2 asm: ~(~A · ~B) ← Assume opposite.

Then we're stuck!

We can't apply S- or I-rules or RAA; and we don't have enough simple wffs for a refutation.

START: Assume the opposite of the conclusion.

1 $\sim(A \cdot B)$

[$\therefore (\sim A \cdot \sim B)$

2 asm: $\sim(\sim A \cdot \sim B)$

3 asm: A {break up 1} ←

When you're stuck, try to make another assumption.

ASSUME: Look for a complex wff that isn't starred or blocked off or broken. This wff will have one of these forms:

NOT-BOTH	$\sim(A \cdot B)$
OR	$(A \vee B)$
IF-THEN	$(A \supset B)$

Assume one side or its negation – and then return to step 2 (S&I).

** 1 ~(A • B)

[∴ (~A • ~B)

2 asm: ~(~A • ~B)

3 asm: A {break up 1}

4 ∴ ~B {from 1 and 3} 🡐 Derive further lines.

We’re stuck again! But now all complex wffs are either starred or blocked off or broken.

S&I: Go through the complex wffs that aren’t starred or blocked off and use these to derive new wffs using S- and I-rules. Star (*with one star for each live assumption*) any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

** 1 $\sim(A^1 \cdot B^0) = 1$

[$\therefore (\sim A^1 \cdot \sim B^0) = 0$

2 asm: $\sim(\sim A \cdot \sim B)$

3 asm: A {break up 1}

4 $\therefore \sim B$ {from 1 and 3}

Invalid

A, ~B

REFUTE: Construct a refutation box if you can't apply S- and I-rules or RAA further, and yet all complex wffs are either starred or blocked off or broken.

* 1 (B ∨ A) Valid
2 (B ⊃ A)
[∴ ~(A ⊃ ~A)
* 3 ┌ asm: (A ⊃ ~A)
4 │ ┌ asm: B {break up 1}
5 │ │ ∴ A {from 2 and 4}
6 │ └ ∴ ~A {from 3 and 5}
7 │ ∴ ~B {from 4; 5 contradicts 6}
8 │ ∴ A {from 1 and 7}
9 └ ∴ ~A {from 3 and 8}
10 ∴ ~(A ⊃ ~A) {from 3; 8 contradicts 9}

** 1 $\sim(A^1 \cdot B^0) = 1$ Invalid
[∴ $(\sim A^1 \cdot \sim B^0) = 0$
2 asm: ~(~A • ~B)
3 asm: A {break up 1}
4 ∴ ~B {from 1 and 3}

A, ~B

Start	S&I	RAA	Assume	Refute

Traditional Copi proofs use eight inference rules and ten replacement rules. Here are the inference rules:

AD Addition	$\dfrac{P}{(P \vee Q)}$
CJ Conjunction	$\dfrac{\begin{array}{c}P \\ Q\end{array}}{(P \cdot Q)}$
DI Dilemma	$\dfrac{\begin{array}{c}((P \supset Q) \cdot (R \supset S)) \\ (P \vee R)\end{array}}{(Q \vee S)}$
DS Disjunctive Syllogism	$\dfrac{\begin{array}{c}(P \vee Q) \\ \sim P\end{array}}{Q}$
HS Hypothetical Syllogism	$\dfrac{\begin{array}{c}(P \supset Q) \\ (Q \supset R)\end{array}}{(P \supset R)}$
MP Modus Ponens	$\dfrac{\begin{array}{c}(P \supset Q) \\ P\end{array}}{Q}$
MT Modus Tollens	$\dfrac{\begin{array}{c}(P \supset Q) \\ \sim Q\end{array}}{\sim P}$
SP Simplification	$\dfrac{(P \cdot Q)}{P}$

Here are the ten Copi replacement rules:

AS	Association	$(P \vee (Q \vee R)) = ((P \vee Q) \vee R)$ $(P \cdot (Q \cdot R)) = ((P \cdot Q) \cdot R)$
CM	Commutation	$(P \vee Q) = (Q \vee P)$ $(P \cdot Q) = (Q \cdot P)$
DB	Distribution	$(P \cdot (Q \vee R)) = ((P \cdot Q) \vee (P \cdot R))$ $(P \vee (Q \cdot R)) = ((P \vee Q) \cdot (P \vee R))$
DM	De Morgan	$\sim(P \cdot Q) = (\sim P \vee \sim Q))$ $\sim(P \vee Q) = (\sim P \cdot \sim Q)$
DN	Double Negation	$P = \sim\sim P$
EQ	Equivalence	$(P \equiv Q) = ((P \supset Q) \cdot (Q \supset P))$ $(P \equiv Q) = ((P \cdot Q) \vee (\sim P \cdot \sim Q))$
EX	Exportation	$((P \cdot Q) \supset R) = (P \supset (Q \supset R))$
IM	Implication	$(P \supset Q) = (\sim P \vee Q)$
RP	Repetition	$P = (P \vee P)$ $P = (P \cdot P)$
TR	Transposition	$(P \supset Q) = (\sim Q \supset \sim P)$

Conclusion: B

1 T
2 $(T \supset (B \vee M))$
3 $(M \supset H)$
4 $\sim H$
5 $(B \vee M)$ {MP 1+2}
6 $\sim M$ {MT 3+4}
7 $(M \vee B)$ {CM 5}
8 B {DS 6+7}

Many Copi proofs directly derive the conclusion from the premises. Copi also provides for conditional and indirect (RAA) proofs.

CP Conditional Proof	If you assume P and later derive Q, then you can star all the lines from P to Q [showing that you aren't to use them to derive further steps] and then derive $(P \supset Q)$.
RA Reductio ad Absurdum	If you assume P and later derive $(Q \cdot \sim Q)$, then you can star all the lines from P to $(Q \cdot \sim Q)$ [showing that you aren't to use them to derive further steps] and then derive $\sim P$.

Truth trees break formulas into the cases that make them true. Here's a truth tree for "(A • ~B), (B ∨ C) ∴ C":

(A • ~B) *

(B ∨ C) *

Conclusion: C **~C**

A

~B

B | C

* | *

Valid: every branch closes

An argument is valid if and only if every branch eventually *closes* (has a self-contradiction).